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Generalised discrete torsion and mirror symmetry for G_2 manifolds

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Abstract: A generalisation of discrete torsion is introduced in which different discrete torsion phases are considered for the different fixed points or twist fields of a twisted sector. The constraints that arise from modular invariance are analysed carefully. As an application we show how all the different resolutions of the $T^7/\mathbb{Z}_2^3$ orbifold of Joyce have an interpretation in terms of such generalised discrete torsion orbifolds. Furthermore, we show that these manifolds are pairwise identified under G_2 mirror symmetry. From a conformal field theory point of view, this mirror symmetry arises from an automorphism of the extended chiral algebra of the G_2 compactification.

Keywords: String Duality, Conformal Field Models in String Theory.

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1. Introduction

It has been known for a long time that orbifolds in string theory are not uniquely characterised in terms of the action of the orbifold group on the states of the original theory. In fact, in order to construct the theory one also has to specify the action of the orbifold group Γ in the various twisted sectors of the theory, and in general this is not unambiguously defined. In particular, as was pointed out by Vafa [1], there is at least the freedom to modify the action of g in the h-twisted sector by a phase $\epsilon(g,h)$. Provided that these phases correspond to a 2-cocycle $H^2(\Gamma, \mathrm{U}(1))$, the resulting theory is modular invariant and thus consistent (given that the original orbifold theory was so). Thus if $H^2(\Gamma, \mathrm{U}(1)) \neq \{e\}$, the orbifold construction is ambiguous, and needs to be specified further.

One may wonder whether the above ambiguity is the only ambiguity that is consistent with modular invariance, or whether there are additional possibilities in general. Clearly, any such additional possibilities will depend on the specifics of the theory in question, and it will therefore not be possible to give a general analysis as for the case of conventional discrete torsion. However, it is nevertheless interesting to understand whether there are such additional theories in specific instances.

Our interest in this problem arose from the analysis of orbifolds that describe the compactification of IIA or IIB string theory on the $\mathrm{G_2}$ manifolds of Joyce [2, 3], in particular the family of nine $\mathrm{G_2}$ manifolds obtained by inequivalent resolutions of the $T^7/\mathbb{Z}_2^3$ orbifold of [4, chapter 12.3]. It was shown in [5] how one of the nine possible resolutions can be constructed in string theory. By switching on (conventional) discrete torsion, another resolution was found in [6, 7], but it was not clear how to obtain the remaining seven in terms of string theory. In this paper we want to explain how these remaining resolutions can be constructed. This involves a generalisation of the usual discrete torsion construction in which different discrete torsion phases are switched on for the different fixed points of a given twisted sector. The constraint that the resulting theory must still be modular invariant imposes some constraints on the choice of these phases, and we shall analyse them in detail. In fact, we shall find that there are (up to some relabelling) precisely nine different string theories that are allowed by these constraints, and that they correspond precisely to the nine different resolutions found by Joyce.

For a closely related Calabi-Yau manifold, it was shown in [8] that the orbifold with discrete torsion is related to the same orbifold without discrete torsion by mirror symmetry [9]. In fact, mirror symmetry simply corresponds to T-duality along three circles in this case. This suggests that something similar may be true for the G_2 manifolds in question. (The idea that some version of mirror symmetry should also apply to G_2 manifolds was first proposed in [3] and argued for on physical grounds in [5].) It was shown in [6, 7] that the theory with and without (conventional) discrete torsion are indeed related by three T-dualities to one another, and this suggests that one should regard them as mirror partners.

For Calabi-Yau manifolds, mirror symmetry can be understood, in terms of the underlying conformal field theory description, as the effect of a non-trivial automorphism of the (right-moving) extended $\mathcal{N} = 2$ superconformal algebra that is always present for Calabi-Yau compactifications [10]. The extended algebra for G_2 compactifications has been constructed in [5] (see also [11]), and one may ask whether mirror symmetry for G_2 manifolds can be similarly interpreted. The G_2 algebra contains a non-trivial automorphism that leaves the $\mathcal{N} = 1$ superconformal subalgebra invariant.[1] By considering three T-dualities along suitable directions one can induce this automorphism on the right-movers (without modifying the left-movers). Depending on which realisation one chooses, this maps the IIA/IIB theory on one of the nine G_2 orbifolds to the IIB/IIA theory on the *same* orbifold, or to IIB/IIA theory on the orbifold where *all discrete torsion phases have been inverted.* In the latter case, the 'mirror map' therefore relates the nine G_2 manifolds pairwise (with one manifold being its own mirror); the former possibility, on the other hand, is the string theory realisation of the symmetry proposed in [14].

This paper is organised as follows. In section 2 we discuss the generalised orbifold construction in the simpler example of the Calabi-Yau compactification on $T^6/\mathbb{Z}_2^2$ for which discrete torsion and mirror symmetry were studied in detail in [8]. We explain how to solve the constraints imposed by modular invariance, and show that there are (up to suitable relabellings) seventeen different theories. Furthermore, we study the effect of mirror symmetry on these theories, and demonstrate that these seventeen theories are pairwise identified by mirror symmetry, with one being its own mirror. From a geometric point of view, the different theories correspond to the different desingularisations that can be chosen for the different orbifold singularities.

[1] This automorphism was already observed in [12], see also [13].

In section 3, the analogous construction is performed for the $T^7/\mathbb{Z}_2^3$ orbifold of Joyce (see [4, chapter 12.3]). In this case, the analysis of modular invariance is more cumbersome, and some of the details are spelled out in the appendix. There are now nine different orbifold theories, and they correspond precisely to the different resolutions of Joyce. We also explain how the mirror automorphism of the G_2 manifold can be implemented for this theory, and how it relates either a G_2 manifold to itself, or to a different G_2 manifold. Finally, section 4 contains our conclusions.

2. Generalised discrete torsion and mirror symmetry for Calabi-Yau 3-folds

In this section we describe how to generalise discrete torsion for the familiar orbifold $X = T^6/\mathbb{Z}_2^2$ that was studied in detail in [8] (see also [15]). As we shall see, the different constructions correspond geometrically to different choices for how to desingularise the various orbifold singularities. Finally we shall discuss how certain T-dualities implement the mirror symmetry, and how this exchanges orbifolds with different choices of discrete torsion.

2.1 Generalised discrete torsion

The Hilbert space $\mathcal{H}$ of an orbifold theory for an abelian orbifold group Γ consists of sectors $\mathcal{H}_h$, one such sector for each element h of the orbifold group Γ. The sector $\mathcal{H}_h$ describes those closed string states that are twisted by the action of h along their spacelike direction, i.e. $x(\tau,\sigma=2\pi) = h\cdot x(\tau,\sigma=0)$. To be explicit, we consider the $T^6/\mathbb{Z}_2^2$ orbifold of [8], where the two generators of $\Gamma = \mathbb{Z}_2\times\mathbb{Z}_2$ act multiplicatively as

$$\begin{array}{c|cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline \alpha & +1 & +1 & -1 & -1 & -1 & -1 \\ \beta & -1 & -1 & +1 & +1 & -1 & -1 \end{array} \tag{2.1}$$

on the coordinates of the torus. We will call I_h^+ and I_h^- the index set of those coordinates on which h acts with even and odd parity, respectively.

Since each non-trivial orbifold group element inverts four of the six directions, each has sixteen fixed points. The twisted sector $\mathcal{H}_h$ can therefore be decomposed into sixteen isomorphic Hilbert spaces $\mathcal{H}_{h;f}$, one for each fixed point f. For each fixed point, the Hilbert space $\mathcal{H}_{h;f}$ is generated by the action of the oscillators from a ground state that is characterised by its momentum and winding number. (In the twisted sector $\mathcal{H}_h$, momentum and winding numbers only exist for the directions I_h^+.) The total space of states of the orbifold theory consists then of the sum of all twisted sectors, where in each twisted sector only the states that are invariant under the action of the orbifold group survive. The complete partition function of the theory is therefore given by

$$Z(q,\bar{q}) = \frac{1}{|\Gamma|}\sum_{g,h\in\Gamma}\mathrm{Tr}_{\mathcal{H}_h}\left(g\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right) =: \frac{1}{|\Gamma|}\sum_{g,h\in\Gamma} {}^{g}\underset{h}{\Box}\,, \tag{2.2}$$

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where the sum over $g \in \Gamma$ implements the projection onto Γ-invariant states. At first this expression is somewhat formal since *a priori* it is not clear how to define the action of g in the h-twisted sector (unless $h = e$, in which case $\mathcal{H}_e$ is just the original space of states).

Suppose now that one has found one consistent action of g in each h-twisted sector that leads to a modular invariant partition function. The idea of discrete torsion is that one can modify this action by a phase $\epsilon(g, h)$,

$$\hat{g}|_{\mathcal{H}_h} := \epsilon(g,h)\; g|_{\mathcal{H}_h} \ , \tag{2.3}$$

where $\epsilon(g, h)$ depends on g and h, but is otherwise the same for *all* states in the sector $\mathcal{H}_h$. In order for the action defined by $\hat{g}$ to define a representation of Γ, we need that

$$\epsilon(g_1\, g_2, h) = \epsilon(g_1, h)\, \epsilon(g_2, h)\,. \tag{2.4}$$

Furthermore, in order for the resulting theory to be modular invariant, one requires [1]

$$\epsilon(g,h) = \epsilon(g^a h^b, g^c h^d)\,, \qquad ad - bc = 1\,. \tag{2.5}$$

Different sets of discrete torsion phases are in one-to-one correspondence with elements in $H^2(\Gamma, \mathrm{U}(1))$.

In the above, we have modified the action of g on $\mathcal{H}_h$ by an overall phase that is the same for all states in the sector $\mathcal{H}_h$. Provided that the phases satisfy (2.4) this clearly defines a consistent action of Γ on $\mathcal{H}_h$. However, in general this is not the only way in which we can modify the action of Γ on $\mathcal{H}_h$. As we have mentioned above, each $\mathcal{H}_h$ is the direct sum of sixteen copies,

$$\mathcal{H}_h = \bigoplus_{f=1}^{16} \mathcal{H}_{h;f}\,, \tag{2.6}$$

where $\mathcal{H}_{h;f}$ describe the h-twisted states associated to the fixed point f. Each of these spaces forms an irreducible representation of the oscillators. Since Γ has a prescribed action on the oscillators, the action of g on a given state in $\mathcal{H}_{h;f}$, determines the action on all of $\mathcal{H}_{h;f}$ uniquely. On the other hand, it does *not* determine the action of g on the states in $\mathcal{H}_{h;f'}$ with $f' \neq f$. Thus, we should be able to choose discrete torsion phases separately for the different fixed point components, i.e. the discrete torsion phase should be allowed to depend on f,

$$\hat{g}|_{\mathcal{H}_{h;f}} := \epsilon_f(g,h)\; g|_{\mathcal{H}_{h;f}} \ . \tag{2.7}$$

In order for this to define a consistent action of Γ on $\mathcal{H}$, each $\epsilon_f(g, h)$ must satisfy (2.4). In addition, we must require that the resulting partition function is still modular invariant. This last condition requires a little bit of care and depends on the specifics of the theory in question. In order to analyse this issue, let us write out the contribution of the various sectors to the partition function (here $g, h \in \{\alpha, \beta, \alpha\beta\}$ and we have only written down the bosonic contributions)

$$\underset{e}{{}^{e}\square} = \mathrm{Tr}_{\mathcal{H}_e}\left(q^{L_0 - c/24}\bar{q}^{\bar{L}_0 - c/24}\right) = \frac{1}{|\eta|^{12}} \sum_{(p_L, p_R)\in\Gamma^{6,6}} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2}\,,$$

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$$
g\,\underset{e}{\square} = \mathrm{Tr}_{\mathcal{H}_e}\left(g\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right) = 16\frac{1}{|\vartheta_2|^4}\sum_{(p_L,p_R)\in\Gamma^{2,2}} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2}\,,
$$

$$
e\,\underset{g}{\square} = \mathrm{Tr}_{\mathcal{H}_g}\left(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right) = \sum_{f=1}^{16}\frac{1}{|\vartheta_4|^4}\sum_{(p_L,p_R)\in\Gamma^{2,2}} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2}\,,
$$

$$
g\,\underset{g}{\square} = \mathrm{Tr}_{\mathcal{H}_g}\left(g\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right) = \sum_{f=1}^{16}\epsilon_f(g,g)\frac{1}{|\vartheta_3|^4}\sum_{(p_L,p_R)\in\Gamma^{2,2}} q^{\frac{1}{2}p_L^2}\bar{q}^{\frac{1}{2}p_R^2}\,,
$$

$$
g\,\underset{h}{\square}\big|_{g\neq h} = \mathrm{Tr}_{\mathcal{H}_h}\left(g\, q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right)\Big|_{g\neq h} = \sum_{f=1}^{16}\epsilon_f(g,h)\,.
$$

Here

$$
\eta = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{3}{2}\left(n-\frac{1}{6}\right)^2}
$$

is the Dedekind η-function, while the ϑ-functions are given by

$$
\vartheta_2 = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}\left(n-\frac{1}{2}\right)^2}\,, \qquad \vartheta_3 = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2}\,, \qquad \vartheta_4 = \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{1}{2}n^2}\,.
$$

The lattice $\Gamma^{6,6}$ is the full momentum and winding lattice of the underlying T^6, which we assume to decompose as $T^6 = T^2\times T^2\times T^2$. As a consequence, $\Gamma^{6,6}$ is a direct sum of three $\Gamma^{2,2}$ lattices that are associated to the three different T^2s. All of these lattices are even and self-dual, and the corresponding partition functions therefore transform in a simple manner under the modular group. The modular invariance of the full partition function then requires that

$$
e\,\underset{g}{\square}(q(\tau+1)) = g\,\underset{g}{\square}(q(\tau)) \qquad \Rightarrow \quad \epsilon_f(g,g) = 1\,, \quad \forall\, g\,, f\,, \tag{2.8}
$$

$$
g\,\underset{h}{\square}\left(q\left(-\frac{1}{\tau}\right)\right) = h\,\underset{g}{\square}(q(\tau)) \qquad \Rightarrow \quad \sum_{f=1}^{16}\epsilon_f(g,h) = \sum_{f=1}^{16}\epsilon_f(h,g)\,. \tag{2.9}
$$

The first of these constraints (that arises from the modular T-transformation) leaves us with 16 signs $\epsilon_{h;f}$, $f = 1,\ldots,16$ for each twisted sector labelled by h. [For example, $\epsilon_{\alpha;f} = \epsilon_f(\beta,\alpha) = \epsilon_f(\alpha\beta,\alpha)$.] The second constraint (that arises from the modular S-transformation) forces the number of positive signs to be the same in each sector. By relabelling the fixed points if necessary, we can therefore set $\epsilon_f := \epsilon_{\alpha;f} = \epsilon_{\beta;f} = \epsilon_{\alpha\beta;f}$.

In [1] it was shown that modular invariance at one-loop, i.e. (2.8) and (2.9), together with (2.4) imply via the factorisation property that the (bosonic) orbifold is modular invariant on any genus n Riemann surface (at least in the limit where the latter degenerates into n 1-tori connected by infinitly long and thin cylinders). For generalised discrete torsion the analogeous argument does not work any more; this is to say, the one-loop modular

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invariance (together with the factorisation property) does not imply automatically that the theory is modular invariant on higher genus surfaces as well. [The problem arises when trying to show that the vacuum amplitude is invariant under the Dehn twist that links adjacent tori.] On the other hand, this does not imply that the theory with generalised discrete torsion does not satisfy higher genus modular invariance; it merely means that higher genus modular invariance will only hold provided that the vacuum amplitudes $A(g_1, h_1, f_1; g_2, h_2, f_2; \ldots; g_n, h_n, f_n)$ satisfy suitable additional properties. [For example, one can show that 2-loop modular invariance follows for suitable assignments of the different discrete torsion phases to the different fixed points provided that the amplitudes $A(g_1, h_1, f_1; g_2, h_2, f_2; \ldots; g_n, h_n, f_n)$ involving 'different' fixed points vanish in the above limit.] It would obviously be very interesting to analyse this question for the theories we discuss here, but unfortunately, this is a very difficult problem which seems to be out of reach at the moment. [It would require constructing the actual 2-loop amplitudes for the various twisted sectors and twists, but such amplitudes have not even be constructed in much simpler examples.] It is therefore conceivable that additional restrictions on the choice of the different discrete torsion phases will be required by higher genus modular invariance.

The contribution of the fermions is also described by ϑ functions, and their inclusions does not destroy the modular invariance properties.[2]

In total there are therefore 17 different choices of discrete torsion given by the number $\ell \in \{0, \ldots, 16\}$ of positive signs among ϵ_f.[3] The extremal cases $\ell = 0$ and $\ell = 16$ correspond precisely to the situation with and without discrete torsion, respectively [8], but now we also have intermediate possibilities.

2.2 Discrete torsion and the resolution of orbifold singularities

In this section we will give a geometrical interpretation of these generalised discrete torsion theories. It is common lore that the untwisted sector captures the geometry of the singular orbifold, whereas the twisted sectors describe their resolution. It is therefore not surprising that discrete torsion has to do with the way in which one resolves the singularities. We will find that the spectrum of ground states in the h-twisted sector depends on discrete torsion if the singularity of h-fixed loci can be resolved in inequivalent ways. A particular choice of discrete torsion then tells us which resolution is chosen. Our generalisation of discrete torsion corresponds thus simply to the possibility of choosing different resolutions for different fixed points.

[2]Strictly speaking, the inclusion of fermions introduces fermionic zero modes into various sectors which in turn make the associated vacuum amplitudes vanish. Formally, the constraints of modular invariance therefore seem to be weaker in the fermionic case. However, these 'accidental' vanishings can be lifted by considering torus amplitudes that include an appropriate number of fermionic zero modes, and it is thus believed that the bosonic conditions are necessary and sufficient to guarantee modular invariance in the fermionic case as well. The inclusion of fermions may on the other hand modify the conditions for modular invariance at higher genus.

[3]Actually, the number of different theories is bigger since theories that differ in the way these signs are qdistributed among the fixed points will in general be different. However, their Betti/Hodge numbers will be the same.

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In order to relate the orbifold CFT to the topology of the target space we exploit the isomorphism between the space of RR ground states and the cohomology of the target space [16]. To this end we accompany the left- and right-moving part of each coordinate x_j with a left- and right-moving (2d) Majorana-Weyl spinor ψ^j and $\tilde{\psi}^j$, respectively. If the original metric on T^6 was chosen to be the flat one, their zero modes satisfy the Clifford algebra

$$\{\psi_0^i, \psi_0^j\} = 2\delta^{ij}\,, \qquad \{\tilde{\psi}_0^i, \tilde{\psi}_0^j\} = 2\delta^{ij}\,, \qquad \{\psi_0^i, \tilde{\psi}_0^j\} = 0\,. \tag{2.10}$$

In order to build the Fock space of physical states we define

$$\psi_\pm^j := \frac{1}{2}\left(\psi_0^j \pm i\tilde{\psi}_0^j\right)\,, \qquad j = 1, \ldots, 6 \tag{2.11}$$

which satisfy the algebra

$$\{\psi_\pm^i, \psi_\mp^j\} = \delta^{ij}\,, \qquad \{\psi_\pm^i, \psi_\pm^j\} = 0\,. \tag{2.12}$$

We can then choose the ψ_+^j to be creators and ψ_-^j to be annihilators,

$$\psi_-^j|0\rangle = 0 \qquad \Rightarrow \qquad \psi_0^j|0\rangle = i\tilde{\psi}_0^j|0\rangle \tag{2.13}$$

in the Fock space. Note that the above is not the standard choice of generators for Calabi-Yau target spaces (see e.g. [17]); however, the above convention makes also sense for spaces of odd dimension (such as the G_2 spaces we shall consider in the next section) and it is therefore more convenient for us than the usual definition which only works for spaces of even dimension. Of course the two choices generate isomorphic Fock spaces.

The orbifold invariant RR ground states in the untwisted sector are then given by

$$\begin{aligned}
&|0\rangle \\
&\psi_+^1\psi_+^2|0\rangle,\quad \psi_+^3\psi_+^4|0\rangle,\quad \psi_+^5\psi_+^6|0\rangle \\
&\psi_+^1\psi_+^3\psi_+^5|0\rangle,\quad \psi_+^1\psi_+^3\psi_+^6|0\rangle,\quad \psi_+^1\psi_+^4\psi_+^5|0\rangle,\quad \psi_+^1\psi_+^4\psi_+^6|0\rangle \\
&\qquad\qquad \psi_+^2\psi_+^3\psi_+^5|0\rangle,\quad \psi_+^2\psi_+^3\psi_+^6|0\rangle,\quad \psi_+^2\psi_+^4\psi_+^5|0\rangle,\quad \psi_+^2\psi_+^4\psi_+^6|0\rangle \\
&\psi_+^1\psi_+^2\psi_+^3\psi_+^4|0\rangle,\quad \psi_+^1\psi_+^2\psi_+^5\psi_+^6|0\rangle,\quad \psi_+^3\psi_+^4\psi_+^5\psi_+^6|0\rangle \\
&\psi_+^1\psi_+^2\psi_+^3\psi_+^4\psi_+^5\psi_+^6|0\rangle\,.
\end{aligned} \tag{2.14}$$

Identifying

$$\psi_+^{j_1}\ldots\psi_+^{j_n}|0\rangle \simeq dx^{j_1}\wedge\ldots\wedge dx^{j_n}$$

these RR ground states become the Γ-invariant harmonic forms on T^6.

Since the three twisted sectors are isomorphic we will discuss the RR ground states of only one of them, say the α-twisted sector. In the RR sector then only ψ^1 and ψ^2 have zero-modes since the twist changes the spin structure of the remaining fermions. Let $|0,0;f\rangle_\alpha$ be one of the 16 highest weight states with vanishing momentum and winding. The Γ-invariant RR ground states built on this state are then

$$|0,0;f\rangle_\alpha \qquad \text{and} \qquad \psi_+^1\psi_+^2|0,0;f\rangle_\alpha \tag{2.15}$$

if $\beta|0,0;f\rangle_\alpha = \alpha\beta|0,0;f\rangle_\alpha = |0,0;f\rangle_\alpha$, i.e. $\epsilon_{\alpha;f} = 1$, or

$$\psi^1_+|0,0;f\rangle_\alpha \qquad \text{and} \qquad \psi^2_+|0,0;f\rangle_\alpha \tag{2.16}$$

if $\beta|0,0;f\rangle_\alpha = \alpha\beta|0,0;f\rangle_\alpha = -|0,0;f\rangle_\alpha$, i.e. $\epsilon_{\alpha;f} = -1$. Again we can identify these states with harmonic forms. The state $|0,0;f\rangle_\alpha$ however is now not identified with the constant zero-form but with the two-form $\omega_{\alpha;f}$ representing the class of the exceptional divisor that arises in the resolution of this singularity. In the complex structure

$$z_1 = x_1 + ix_2\,, \qquad z_2 = x_3 + ix_4\,, \qquad z_3 = x_5 + ix_6\,,$$

this form will be of type (1,1). Using the same complex structure to complexify the fermions, the highest weight sector $|0,0;f\rangle_\alpha$ contributes one class to $h^{1,1}$ and $h^{2,2}$ for $\epsilon_{\alpha;f} = 1$, and one class to $h^{2,1}$ and $h^{1,2}$ for $\epsilon_{\alpha;f} = -1$.

Let $\ell \in \{0,\dots,16\}$ be the number of positive signs among ϵ_f, then since all three twisted sectors are isomorphic their total contribution to the cohomology of the target space is

$$h^{p,q}(\text{twisted sectors}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 3\ell & 48-3\ell & 0 \\ 0 & 48-3\ell & 3\ell & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.17}$$

where p and q label the rows and columns respectively. In total the 17 different choices of discrete torsion lead to target manifolds X_ℓ with Hodge diamonds

$$h^{p,q}(X_\ell) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 3(\ell+1) & 51-3\ell & 0 \\ 0 & 51-3\ell & 3(\ell+1) & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad \text{for } \ell \in \{0,\dots,16\}\,. \tag{2.18}$$

Next we will show that the choice of the discrete torsion phase at a fixed point is associated with the choice of how the corresponding singularity is resolved. Consider for definiteness the complex codimension two singularities due to the action of α. Locally each of these 16 singularities inside T^6/Γ looks like

$$(T^2 \times \mathbb{C}^2/\{\pm 1\})/\langle \beta, \alpha\beta\rangle\,. \tag{2.19}$$

As is explained in [4], there are two inequivalent ways to desingularise the $\mathbb{C}^2/\{\pm1\}$ singularity inside the total orbifold.

The first one is the blow up Y_1 of $\mathbb{C}^2/\{\pm1\} = (z_2,z_3)/\{\pm1\}$ at the origin, which creates an exceptional divisor $\Sigma_1 = \mathbb{C}P^1 \simeq [z_2,z_3] \subset Y_1$ whose homology class generates $H_2(Y_1,\mathbb{R}) = \mathbb{R}$. The actions of β and $\alpha\beta$ lift to Σ_1 and act as

$$\beta : [z_2,z_3] \mapsto [z_2,-z_3]\,,$$
$$\alpha\beta : [z_2,z_3] \mapsto [-z_2,z_3]\,,$$

which both preserve the orientation of Σ_1 and hence the induced maps β_* and $\alpha\beta_*$ on $H_2(Y_1,\mathbb{R})$ are the identity.

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The second way to desingularise $\mathbb{C}^2/\{\pm 1\}$ is to deform it. To this end one defines $\sigma : \mathbb{C}^2/\{\pm 1\} \to \mathbb{C}^3$ by

$$\sigma \ : \ \pm (z_2, z_3) \mapsto (z_2^2 - z_3^2, iz_2^2 + iz_3^2, 2z_2z_3)\,, \tag{2.20}$$

which identifies $\mathbb{C}^2/\{\pm 1\}$ with the quadratic

$$\{(w_1, w_2, w_3) \in \mathbb{C}^3 \mid w_1^2 + w_2^2 + w_3^2 = 0\}\,.$$

Let $\eta \in \mathbb{C}^\times$ be small and non-zero and

$$Y_2 := \{(w_1, w_2, w_3) \in \mathbb{C}^3 \mid w_1^2 + w_2^2 + w_3^2 = \eta\}\,, \tag{2.21}$$

then Y_2 is a smoothing of $\mathbb{C}^2/\{\pm 1\}$ that is diffeomorphic to Y_1. Let $\eta = re^{2i\phi}$ with $r \in \mathbb{R}$ positive and $\phi \in [0, \pi)$, and define

$$\Sigma_2 := \{(e^{i\phi}x_1, e^{i\phi}x_2, e^{i\phi}x_3) \mid x_j \in \mathbb{R},\ x_1^2 + x_2^2 + x_3^2 = r\}\,, \tag{2.22}$$

then the homology class of $\Sigma_2 \simeq S^2 \subset Y_2$ generates $H_2(Y_2, \mathbb{R}) = \mathbb{R}$. Since β and $\alpha\beta$ both act as

$$\beta, \alpha\beta \ : \ (w_1, w_2, w_3) \mapsto (w_1, w_2, -w_3)\,,$$

they preserve Σ_2 but reverse its orientation. Thus their induced maps β_* and $\alpha\beta_*$ on $H_2(Y_2, \mathbb{R})$ are minus the identity.

Since the exceptional divisor $\Sigma_{\alpha;f}$ (generating $H_2(Y_1, \mathbb{R})$ or $H_2(Y_2, \mathbb{R})$, respectively) corresponds to the ground state $|0, 0; f\rangle_\alpha$ of the corresponding twisted sector, the discrete torsion signs $\epsilon_{\alpha;f}$ are geometrically just the eigenvalues of β_* (and $\alpha\beta_*$) on the homology classes of the exceptional divisors $\Sigma_{\alpha;f}$. [The fact that these eigenvalues appear in the partition function is due to the contribution of the B-field evaluated on these classes; for a geometrical description of discrete torsion as a choice of representation of the orbifold group on the B-field see [18].] Hence the parameter ℓ counts how many of the 16 singularities of complex codimension two generated by α we have chosen to blow up, instead of deforming it. The analysis is obviously identical for the sectors twisted by β and $\alpha\beta$. The cohomology of the resulting space is then exactly the one given in (2.18).

From this geometric point of view, it is *a priori* not clear why ℓ has to be taken to be the same for all three twisted sectors. (This is the condition that arose from the requirement that the partition function is invariant under the modular S-transformation.) However, the reason may be related to the fact that the orbifold has an S_3-symmetry of permutations of z_1, z_2, z_3 that also permutes the three twisted sectors. If this symmetry is to be respected by the discrete torsion phases, then ℓ indeed has to be the same for all three twisted sectors.

2.3 Discrete torsion and mirror symmetry of orbifold Calabi-Yau 3-folds

Finally we want to show that, in this example, T-duality on three of the coordinates generates mirror symmetry and that this exchanges the orbifold with discrete torsion parameter ℓ with the one with parameter $16 - \ell$ in which all the discrete torsion signs are reversed. This is a generalisation of the result of [8], where this was shown by other means for $\ell = 0$.

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The chiral algebra of a string on a Calabi-Yau 3-fold [10] consists of the $\mathcal{N} = 2$ superconformal algebra generated by the stress energy tensor $T_{\rm CY}$, the two supercurrents $G_{\rm CY}, G'_{\rm CY}$ and the U(1)-current J that is extended by a complex current Ω of conformal weight $h_\Omega = 3/2$ and its superpartner $\Psi := \{G_{\rm CY}, \Omega\}$. For the above orbifold theory and with our choice of complex structure, the relevant currents look like

$$T_{\rm CY} = \frac{1}{2}\sum_{j=1}^{6} :\partial x_j \partial x_j: -\frac{1}{2}\sum_{j=1}^{6} :\psi^j \partial\psi^j: , \qquad G_{\rm CY} = \sum_{j=1}^{6} :\psi^j \partial x_j: ,$$
$$G'_{\rm CY} = \sum_{j=1}^{3} \left(\psi^{2j-1}\partial x_{2j} - \psi^{2j}\partial x_{2j-1}\right) , \qquad J = \sum_{j=1}^{3} \psi^{2j-1}\psi^{2j}$$

and

$$\begin{aligned}\Omega = {}& \psi^1\psi^3\psi^5 - \psi^1\psi^4\psi^6 - \psi^2\psi^3\psi^6 - \psi^2\psi^4\psi^5 + \\ &+ i\left(\psi^1\psi^3\psi^6 + \psi^1\psi^4\psi^5 + \psi^2\psi^3\psi^5 - \psi^2\psi^4\psi^6\right) ,\end{aligned} \tag{2.23}$$

which are all preserved by the orbifold action (2.1). There is an isomorphic right-moving chiral algebra in which all fields/operators are replaced by their right-moving partners, e.g. $\psi^j \to \tilde{\psi}^j$.

This chiral algebra has two interesting automorphisms [11]. The first one is a simultaneous phase rotation of the extension operators

$$\Omega \mapsto e^{i\phi}\Omega \qquad \text{and} \qquad \Psi \mapsto e^{i\phi}\Psi . \tag{2.24}$$

The second one is the mirror automorphism

$$\text{mirror}_{\rm CY} \;:\; G'_{\rm CY} \mapsto -G'_{\rm CY} , \quad J \mapsto -J , \quad \Omega \mapsto \Omega^* , \quad \Psi \mapsto \Psi^* , \tag{2.25}$$

with the remaining operators (in particular the $\mathcal{N} = 1$ superconformal subalgebra spanned by $T_{\rm CY}$ and $G_{\rm CY}$) being invariant.

Mirror symmetry for Calabi-Yau manifolds was discovered [9] as a consequence of applying the second automorphism to one of the chiral algebras, say the right-moving one (with tildes). Using the free field representation of the algebra operators, and combining the two automorphisms (with the phase of the former being equal to $e^{i\phi} = \pm 1$) we see that T-duality on three coordinates $x_{j_1}, x_{j_2}, x_{j_3}$ with

$$(j_1, j_2, j_3) \in \{(1,3,5), (1,4,6), (2,3,6), (2,4,5), (1,3,6), (1,4,5), (2,3,5), (2,4,6)\} \tag{2.26}$$

generates the mirror automorphism on the right-moving chiral algebra while leaving the left-moving one invariant. [Note that the combinations of labels are precisely those that appear in the eight terms of Ω, (2.23).] This works because T-duality on x_j leaves the left-moving current ∂x_j invariant, but reverses the right-moving one $\overline{\partial} x_j$. The same then holds for the worldsheet superpartners, i.e. $\psi^j \mapsto \psi^j$ but $\tilde{\psi}^j \mapsto -\tilde{\psi}^j$. Hence T-duality on these three coordinates relates mirror manifolds to one other.

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Next we want to show that T-duality on these three coordinates relates target spaces to each other that correspond to orbifolds where all the discrete torsion signs are reversed. To this end consider the h-twisted sector in the RR sector where only the fermions ψ^j and $\tilde{\psi}^j$ with $j \in I_h^+$ have zero-modes. For definiteness, let us take $h = \alpha$, for which $I_\alpha^+ = \{1, 2\}$; the analysis for the other two cases is identical. We are interested in how the action of β is modified by the T-duality transformation. Since β acts as

$$\beta\psi_0^i\beta = -\psi_0^i\,, \qquad \beta\tilde{\psi}_0^i\beta = -\tilde{\psi}_0^i\,, \qquad i \in I_\alpha^+ = \{1, 2\} \tag{2.27}$$

on the fermionic zero modes, we can represent it on the RR ground states of $\mathcal{H}_{\alpha;f}$ as

$$\beta = \frac{1}{4}\psi_0^1\psi_0^2\tilde{\psi}_0^1\tilde{\psi}_0^2 \cdot \epsilon_{\alpha;f}\,. \tag{2.28}$$

Under any of the above T-duality transformations in (2.26), β then changes sign. Since the above analysis applies uniformly for all fixed points, this operation therefore corresponds to changing all the $\epsilon_{\alpha;f}$. Thus we conclude that mirror symmetry inverts all the discrete torsion signs in this case; in particular, it therefore relates the orbifold labelled by ℓ, X_ℓ, to that labelled by $16 - \ell$, $X_{16-\ell}$. On the Hodge numbers (2.18) this generates indeed the correct symmetry since $h_{1,1}(X_\ell) = h_{2,1}(X_{16-\ell})$.

3. Generalised discrete torsion and mirror symmetry for G_2 orbifolds

In this section we will apply the same reasoning to the orbifold $T^7/\mathbb{Z}_2^3$ of [4, 5] associated to riemannian manifolds of holonomy G_2. Conceptually everything is as in the previous section. However, an interesting application will be mirror symmetry for G_2 manifolds that we consider in section 3.2. But before doing so, let us present the model that we are going to work with.

3.1 Compact orbifold G_2 manifolds and discrete torsion

Consider the orbifold of ([4], chapter 12.3),

$$Y = \frac{T^7}{\mathbb{Z}_2^3}\,, \tag{3.1}$$

where $\Gamma \equiv \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by

$$\begin{aligned}
\alpha &\equiv [(-1,-1,-1,-1,\ 1,\ 1,\ 1)\,;\,(\ 0\ ,\ \ 0\ ,\ 0,\ 0,\ 0,\ 0,\ 0)]\,,\\
\beta &\equiv [(-1,-1,\ 1,\ 1,-1,-1,\ 1)\,;\,(\ 0\ ,\ 1/2,\ 0,\ 0,\ 0,\ 0,\ 0)]\,,\\
\gamma &\equiv [(-1,\ 1,-1,\ 1,-1,\ 1,-1)\,;\,(1/2,\ \ 0\ ,\ 0,\ 0,\ 0,\ 0,\ 0)]\,.
\end{aligned} \tag{3.2}$$

Here the entries in the first vector denote the eigenvalue of the coordinates x_j under the multiplicative group action, $x_j \mapsto \pm x_j$, and the entries of the second vector denote shifts $x_j \mapsto x_j + \epsilon$. The action of γ on x_1 for example is $\gamma : x_1 \mapsto -x_1 + 1/2$. Moreover, the coordinates are taken to have unit periodicity, $x_j \equiv x_j + 1$.

The elements $\alpha\beta, \alpha\gamma, \beta\gamma, \alpha\beta\gamma$ of Γ have no fixed points on T^7 due to the shifts in the first or second coordinate. The fixed points of α, β, γ in T^7 are each 16 copies of T^3, where $\langle\beta,\gamma\rangle$ acts freely on the 16 α-fixed T^3s and $\langle\alpha,\gamma\rangle$ acts freely on the 16 β-fixed T^3s, building four orbits of four tori in each case. However, $\langle\alpha,\beta\rangle$ does not act freely on the set of 16 γ-fixed tori, since $\alpha\beta$ acts trivially (while α and β build eight orbits of order two). The singular set of T^7/Γ is thus a disjoint union of eight copies of T^3 and eight copies of $T^3/\mathbb{Z}_2$. The singularity at each T^3 is locally modelled on $T^3\times\mathbb{C}^2/\{\pm1\}$ whereas the one at each $T^3/\mathbb{Z}_2$ is modelled on $(T^3\times\mathbb{C}^2/\{\pm1\})/\langle\alpha\beta\rangle$. The resolution of the latter is not unique in the same way as in the Calabi-Yau case, due to the different action of $\alpha\beta_*$ on the exceptional divisors arising from the blow up or the deformation of $\mathbb{C}^2/\{\pm1\}$. In fact the analysis can be taken over word by word from the previous section. If $\ell\in\{0,\ldots,8\}$ denotes the number of $T^3/\mathbb{Z}_2$ singularities that we choose to blow up, then we generate nine topologically different manifolds Y_ℓ with Betti-numbers

$$(b_0,\ldots,b_7)(Y_\ell)=(1,0,8+\ell,47-\ell,47-\ell,8+\ell,0,1)\,,\qquad \text{for } \ell=0,1,\ldots,8\,. \tag{3.3}$$

Joyce [4] has shown that all of these are compact G_2 manifolds.

Next we will discuss how the above nine classes of G_2 manifolds are in one-to-one correspondence with nine choices of generalised discrete torsion for this orbifold. As before, we introduce discrete torsion phases for the various twisted sectors, and allow them to be different for the different fixed points or twist fields.[4] For each f and h, the phases $\epsilon_f(g,h)$ must form a representation of Γ w.r.t. the first argument. Furthermore, some of these phases are spurious in that they can be absorbed into the normalisation of the different states. This analysis is discussed in detail in appendix A. After these considerations have been taken into account, we are left with eight signs $\epsilon_{\gamma;\tilde f}=\pm1$ for $\tilde f=1,\ldots,8$, in the γ-twisted sector; another eight signs $\epsilon_{\alpha\beta\gamma;\tilde f}=\pm1$ for $\tilde f=1,\ldots,8$, in the $\alpha\beta\gamma$-twisted sector; and sixteen signs $\epsilon_{\alpha\beta;\tilde f}=\pm1$ for $\tilde f=1,\ldots,16$, in the $\alpha\beta$-twisted sector. However, as in the Calabi-Yau case the constraint from the modular S-transformation relates their sums to each other

$$2\sum_{\tilde f=1}^{8}\epsilon_{\gamma;\tilde f}=\sum_{\tilde f=1}^{16}\epsilon_{\alpha\beta;\tilde f}=2\sum_{\tilde f=1}^{8}\epsilon_{\alpha\beta\gamma;\tilde f}\,. \tag{3.4}$$

By relabelling the twist fields if necessary we can therefore set $\epsilon_{\gamma;\tilde f}=\epsilon_{\alpha\beta\gamma;\tilde f}=\epsilon_{\alpha\beta;\tilde f}=\epsilon_{\alpha\beta;\tilde f+8}$ for $\tilde f=1,\ldots,8$. Up to the ambiguity in how to distribute the various signs among the different twist fields, there are therefore nine different theories which are parametrised by the number $\ell\in\{0,\ldots,8\}$ of plus signs among the eight signs $\epsilon_{\gamma;\tilde f}$.

Next we want to show that these nine choices of discrete torsion correspond indeed to the nine topological classes of G_2 manifolds (3.3). For this we have to look at the RR ground states. Such lowest energy states exist only in the sectors $\mathcal{H}_e,\mathcal{H}_\alpha,\mathcal{H}_\beta$ and $\mathcal{H}_\gamma$ since in the remaining sectors one winding mode takes values in $\mathbb{Z}+\frac{1}{2}$.

[4] Strictly speaking, for each nontrivial $h\in\Gamma$, the different subspaces for which separate discrete torsion phases can be introduced are not labelled by the fixed points of the action of h, but by the twist fields of lowest conformal dimension that generate the irreducible representations of the oscillators as in (2.6). In our orbifold there are 16 such fields for any nontrivial h, even if h does not have any fixed points.

Using the same definitions as in (2.10)–(2.13), the Γ-invariant RR ground states in the untwisted sector $\mathcal{H}_e$ are

$$|0\rangle\,, \quad \psi_+^{j_1}\psi_+^{j_2}\psi_+^{j_3}|0\rangle\,, \quad \psi_+^{j_1}\psi_+^{j_2}\psi_+^{j_3}\psi_+^{j_4}|0\rangle\,, \quad \text{and} \quad \psi_+^1\ldots\psi_+^7|0\rangle\,, \tag{3.5}$$

where the 3-tupel and 4-tupel of indices take values in

$$\begin{aligned}(j_1,j_2,j_3) \in&\{(1,3,6),(1,4,5),(2,3,5),(2,4,6),(1,2,7),(3,4,7),(5,6,7)\}\,,\\(j_1,j_2,j_3,j_4) \in&\{(2,4,5,7),(2,3,6,7),(1,4,6,7),(1,3,5,7),(3,4,5,6),(1,2,5,6),(1,2,3,4)\}\,.\end{aligned}$$

Hence upon the identification (2.2), the untwisted sector contributes just the Γ-invariant harmonic forms of T^7, giving rise to the Betti numbers

$$(b_0^{(e)},\ldots,b_7^{(e)}) = (1,0,0,7,7,0,0,1)\,. \tag{3.6}$$

Next we need to analyse the contributions from the twisted sectors. Since only the sectors $h = \alpha,\beta,\gamma$ have fixed points, only the h-twisted sectors with $h = \alpha,\beta,\gamma$ give rise to massless states. Let us first consider the h-twisted sectors with $h \in \{\alpha,\beta\}$. The action of Γ groups the 16 highest weight states $|0,0;f\rangle_h$ with $f = 1,\ldots,16$ into 4 independent Γ-invariant linear combinations $|0,0;\hat{f}\rangle_h$ for $\hat{f} = 1,\ldots,4$. Geometrically they correspond to the exceptional divisors that resolve the four $T^3 \times \mathbb{C}^2/\{\pm1\}$ singularities that the action of h produces in T^7/Γ. The RR ground states in $\mathcal{H}_h$ are then

$$\begin{aligned}&|0,0;\hat{f}\rangle_h\,,\\&\psi_+^{j_1}|0,0;\hat{f}\rangle_h\,, \quad \psi_+^{j_2}|0,0;\hat{f}\rangle_h\,, \quad \psi_+^{j_3}|0,0;\hat{f}\rangle_h\,,\\&\psi_+^{j_1}\psi_+^{j_2}|0,0;\hat{f}\rangle_h\,, \quad \psi_+^{j_1}\psi_+^{j_3}|0,0;\hat{f}\rangle_h\,, \quad \psi_+^{j_2}\psi_+^{j_3}|0,0;\hat{f}\rangle_h\,,\\&\psi_+^{j_1}\psi_+^{j_2}\psi_+^{j_3}|0,0;\hat{f}\rangle_h\,,\end{aligned}$$

where $\hat{f} = 1,\ldots,4$ and $(j_1,j_2,j_3) = (5,6,7)$ or $(3,4,7)$ for $h = \alpha$ or β, respectively, label the untwisted directions $j_i \in I_h^+$. Provided we choose appropriate relative signs between the four fixed points that correspond to each $\hat{f}$, all of these states are invariant under Γ. Identifying $|0,0;\hat{f}\rangle_h$ with the harmonic two-form $\omega_{h;\hat{f}}$ of the exceptional divisor $\Sigma_{h;\hat{f}}$, these ground states correspond to the harmonic forms on the h-fixed T^3 wedge $\omega_{h;\hat{f}}$, and they contribute the Betti numbers

$$(b_0^{(h)},\ldots,b_7^{(h)}) = (0,0,4,12,12,4,0,0)\,, \qquad \text{for } h = \alpha,\beta\,. \tag{3.7}$$

This leaves us with the (more interesting) γ-twisted sector. Here the action of Γ groups the 16 highest weight states $|0,0;f\rangle_\gamma$ with $f = 1,\ldots,16$ into 8 independent Γ-invariant linear combinations $|0,0;\tilde{f}\rangle_\gamma$ for $\tilde{f} = 1,\ldots,8$. These correspond geometrically to the exceptional divisors that resolve the eight $(T^3 \times \mathbb{C}^2/\{\pm1\})/\langle\alpha\beta\rangle$ singularities that the action of γ produces in T^7/Γ. The $\alpha\beta$-parity of $|0,0;\tilde{f}\rangle_\gamma$ is given by the discrete torsion sign $\epsilon_{\gamma;\tilde{f}}$, and can be chosen to be ±1 independently for each of the eight $\tilde{f}$. Geometrically this corresponds to blowing up the $\mathbb{C}^2/\{\pm1\}$ singularity or deforming it, respectively. If we choose $\epsilon_{\gamma;\tilde{f}} = 1$ (the blow up), then this singularity contributes the RR ground states

$$|0,0;\tilde{f}\rangle_\gamma\,, \quad \psi_+^2|0,0;\tilde{f}\rangle_\gamma\,, \quad \psi_+^4\psi_+^6|0,0;\tilde{f}\rangle_\gamma \quad \text{and} \quad \psi_+^2\psi_+^4\psi_+^6|0,0;\tilde{f}\rangle_\gamma \tag{3.8}$$

and Betti numbers

$$\left(b_0^{(\gamma)}, \ldots, b_7^{(\gamma)}\right)_{\tilde{f}} = (0,0,1,1,1,1,0,0)\,. \tag{3.9}$$

[Again, provided we choose the appropriate relative sign between the two fixed points corresponding to a given $\tilde{f}$, all of the states in (3.8) are invariant under the whole orbifold group Γ.] However, if we choose $\epsilon_{\gamma;\tilde{f}} = -1$ (the deformation), then this singularity contributes the RR ground states

$$\psi_+^4|0,0;\tilde{f}\rangle_\gamma\,, \quad \psi_+^6|0,0;\tilde{f}\rangle_\gamma\,, \quad \psi_+^2\psi_+^4|0,0;\tilde{f}\rangle_\gamma \quad \text{and} \quad \psi_+^2\psi_+^6|0,0;\tilde{f}\rangle_\gamma \tag{3.10}$$

and Betti numbers

$$(b_0^{(\gamma)}, \ldots, b_7^{(\gamma)})_{\tilde{f}} = (0,0,0,2,2,0,0,0)\,. \tag{3.11}$$

If we denote by $\ell \in \{0, \ldots, 8\}$ the number of positive signs among the $\epsilon_{\gamma;\tilde{f}}$ then summing up (3.6), (3.7), (3.9) and (3.11) gives precisely the nine topological classes (3.3) of G_2 manifolds found by Joyce.

3.2 Mirror symmetry for G_2 manifolds

In this subsection we will show that T-duality on three suitably chosen coordinates generates a nontrivial automorphism of one, say the right-moving, extended chiral algebra of the G_2 compactification. In analogy to the Calabi-Yau case we call this automorphism the 'mirror automorphism'. The two theories related by this T-duality are thus physically equivalent. As we shall show, depending on the specific choice of the coordinates, this transformation either reverses *all* discrete torsion signs, or *none*. In the former case, the corresponding mirror map then relates IIA/IIB string theory on Y_ℓ to IIB/IIA string theory on $Y_{8-\ell}$; this is the generalisation of the mirror symmetry mentioned in [6, 7] to $\ell \neq 0$. In the second case, the mirror map relates IIA/IIB string theory on Y_ℓ to IIB/IIA string theory on the same manifold Y_ℓ; this is the mirror symmetry suggested in [14]. It is very satisfying that both these mirror symmetries have an interpretation in terms of the automorphism of the extended G_2 algebra.

The extended chiral algebra of a string moving on a compact G_2 manifold [5] consists of an $\mathcal{N} = 1$ superconformal algebra generated by the stress energy tensor T and the supercurrent G, that is extended by a real current Φ of conformal weight $h_\Phi = 3/2$, a current X of conformal weight $h_X = 2$ and their superpartners $K = \{G, \Phi\}$ and $M = [G, X]$ respectively. The operator Φ corresponds to the 3-form defining the G_2 structure on the target space. In our free field representation, the relevant currents look like

$$\begin{aligned} T &= \frac{1}{2}\sum_{j=1}^{7} :\partial x_j \partial x_j: - \frac{1}{2}\sum_{j=1}^{7} :\psi^j \partial \psi^j:\,, \qquad G = \sum_{j=1}^{7} :\psi^j \partial x_j:\,, \\ \Phi &= \psi^1\psi^3\psi^6 + \psi^1\psi^4\psi^5 + \psi^2\psi^3\psi^5 - \psi^2\psi^4\psi^6 + \psi^1\psi^2\psi^7 + \psi^3\psi^4\psi^7 + \psi^5\psi^6\psi^7\,, \end{aligned} \tag{3.12}$$

and

$$X = -\psi^2\psi^4\psi^5\psi^7 - \psi^2\psi^3\psi^6\psi^7 - \psi^1\psi^4\psi^6\psi^7 + \psi^1\psi^3\psi^5\psi^7$$

$$- \psi^3\psi^4\psi^5\psi^6 - \psi^1\psi^2\psi^5\psi^6 - \psi^1\psi^2\psi^3\psi^4 - \frac{1}{2}\sum_{j=1}^{7} :\psi^j\partial\psi^j: , \tag{3.13}$$

which are all preserved by the orbifold action (3.2). The algebra they satisfy has been worked out in [5]. Of course there is again an isomorphic right-moving algebra.

This extended chiral algebra has two automorphisms. The first is the fermion parity $\psi^j \mapsto -\psi^j$ under which the operators have the eigenvalues

$$\begin{array}{c|cccccc} & T & G & \Phi & X & K & M \\ \hline \text{fermion parity} & + & - & - & + & + & - \end{array} . \tag{3.14}$$

The other automorphism is more interesting,

$$\begin{array}{c|cccccc} & T & G & \Phi & X & K & M \\ \hline \text{mirror}_{G_2} & + & + & - & + & - & + \end{array} . \tag{3.15}$$

It leaves the $\mathcal{N} = 1$ superconformal subalgebra generated by T and G invariant but reverses the operator Φ and its superpartner K. It is the natural analogue of the mirror automorphism of the Calabi-Yau algebra (2.25), and we shall therefore call it the *mirror automorphism.*

For one class of compact G_2 manifolds, namely the manifolds $Y = (\mathrm{CY}_3 \times S^1)/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ acts as a real structure on CY_3 and as an inversion $x_7 \mapsto -x_7$ on the circle, the G_2 mirror automorphism is actually generated by the CY mirror automorphism (applied to the CY_3 part of the above space). To see this, one expresses the generators of the extended chiral algebra for the G_2 compactification in terms of those of the Calabi-Yau manifold and the S^1-compactification (where the latter are described by ∂x^7 and ψ^7) [11],

$$\begin{aligned}
&T = T_{\mathrm{CY}} + \frac{1}{2} :\partial x_7 \partial x_7: - \frac{1}{2} :\psi^7 \partial\psi^7: , \qquad G = G_{\mathrm{CY}} + :\psi^7 \partial x_7: , \\
&\Phi = \mathrm{Im}(\Omega) + :J\psi^7: , \qquad X =: \mathrm{Re}(\Omega)\psi^7: + \frac{1}{2} :JJ: - \frac{1}{2} :\partial\psi^7\,\psi^7: , \\
&K = \mathrm{Im}(\Psi) + :J\partial x_7: + :G'_{\mathrm{CY}}\psi^7: , \\
&M =: \mathrm{Re}(\Psi)\psi^7: - :\mathrm{Re}(\Omega)\partial x_7: + :\partial x_7 \partial\psi^7: + :JG'_{\mathrm{CY}}: - \frac{1}{2}\partial G_{\mathrm{CY}} .
\end{aligned}$$

It is then easy to see that the application of the automorphism (2.25) to the Calabi-Yau generators gives rise to the automorphism (3.15) on the G_2 generators.

Next let

$$\mathcal{I}_3^+ = \{(2,4,6), (2,3,5), (1,2,7)\} , \tag{3.16}$$
$$\mathcal{I}_3^- = \{(1,3,6), (1,4,5), (3,4,7), (5,6,7)\} \tag{3.17}$$

with $\mathcal{I}_3 = \mathcal{I}_3^+ \cup \mathcal{I}_3^-$ be the index set appearing in (3.12). Then since T-duality on the coordinate x_j reverses the right-moving currents $\overline{\partial} x_j$ and $\tilde{\psi}^j$ but leaves the left-moving currents ∂x_j and ψ^j invariant, we see that simultaneous T-duality on $x_{j_1}, x_{j_2}, x_{j_3}$ for $(j_1, j_2, j_3) \in \mathcal{I}_3$

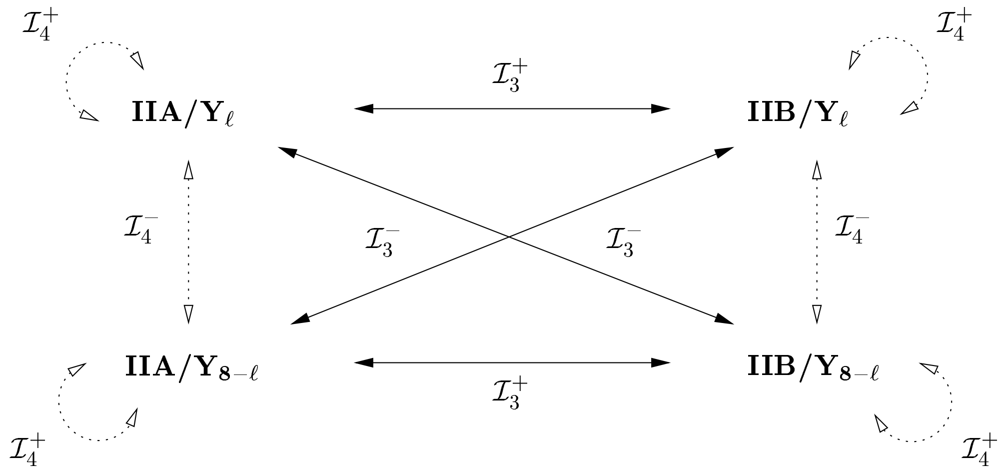

Figure 1: Dualities generated by T-duality on the coordinates of $\mathcal{I}^{\pm}_{3,4}$.

generates the mirror automorphism (3.15) on the right-moving chiral algebra while being the identity on the left-moving one.[5]

Next we want to show that for $(j_1, j_2, j_3) \in \mathcal{I}_3^+$ the T-dualities leave all discrete torsion phases invariant, while for $(j_1, j_2, j_3) \in \mathcal{I}_3^-$ the T-dualities reverse all eight discrete torsion phases. In the former case, this therefore maps the manifold Y_ℓ to itself, whereas in the second case it exchanges Y_ℓ with $Y_{8-\ell}$ (while in both cases exchanging type-IIA and type-IIB strings).

In order to see this, we need to study how the action of these T-dualities modifies the action of $\alpha\beta$ in the γ-twisted RR sector. [This is one of the places where the discrete torsion signs appear; it is not difficult to see that all other sectors behave accordingly.] For the γ-twisted RR sector we have fermionic zero modes for $i = 2, 4, 6$; of these, $\alpha\beta$ inverts the directions $i = 4, 6$. On the RR ground states in $\mathcal{H}_{\gamma;\tilde{f}}$, it can thus be represented in terms of fermionic zero modes as

$$\alpha\beta = \frac{1}{4}\psi_0^4\psi_0^6\tilde{\psi}_0^4\tilde{\psi}_0^6 \cdot \epsilon_{\gamma;\tilde{f}}\,. \tag{3.18}$$

T-duality in the direction j introduces a sign for $\tilde{\psi}_0^j$, but none for ψ_0^j. By inspection of (3.16) and (3.17) it then follows that the T-dualities associated to $\mathcal{I}_3^+$ do not modify the action of $\alpha\beta$, while those associated to $\mathcal{I}_3^-$ do. Since this analysis applies uniformly to all fixed points, it follows that in the second case *all* discrete torsion signs are reversed, and thus that the corresponding duality relates Y_ℓ to $Y_{8-\ell}$. These dualities are summarised in figure 1.

Obviously, we can also combine two (distinct) such transformations; the resulting T-duality transformation inverts then precisely four coordinates. These T-duality transformations fall naturally into two classes $\mathcal{I}_4^\pm$, depending on whether the two T-dualities in

[5]It is easy to check that, apart from those index sets appearing in (3.16) and (3.17), the only other T-duality transformation which has this effect is the T-duality transformation on all seven coordinates. For the following discussion it behaves as an element in $\mathcal{I}_3^+$.

$\mathcal{I}_3$ lie both in the same set $\mathcal{I}_3^{\pm}$, or in different sets,

$$\mathcal{I}_4^+ = \{(1,3,5,7),(1,4,6,7),(3,4,5,6)\}\,, \tag{3.19}$$

$$\mathcal{I}_4^- = \{(2,4,5,7),(2,3,6,7),(1,2,5,6),(1,2,3,4)\}\,. \tag{3.20}$$

All of these transformations obviously leave both chiral algebras invariant, but the transformations in $\mathcal{I}_4^-$ invert all discrete torsion signs, while those in $\mathcal{I}_4^+$ do not modify them. As a consequence, these transformations then relate IIA/IIB theory on Y_ℓ to IIA/IIB theory on Y_ℓ (in the case of $\mathcal{I}_4^+$) or on $Y_{8-\ell}$ (in the case of $\mathcal{I}_4^-$). These dualities (which are a direct consequence of the mirror symmetries associated to $\mathcal{I}_3$) are also summarised in figure 1. Some of them were considered before in [6, 7], and they may be related to the mirror symmetries suggested in [19].

4. Conclusions

In this paper we have shown how all nine G_2 manifolds of Joyce coming from the resolution of $T^7/\mathbb{Z}_2^3$ can be realised in terms of a string theoretic orbifold construction. This involved a generalisation of discrete torsion where the action of g in the h-twisted sector is not just modified by an overall phase, but by phases that are in general different for states that are associated to different fixed points (or twist fields) in the h-twisted sector. We have shown that the resulting theories are still modular invariant at one loop provided that these generalised discrete torsion phases satisfy certain constraints (that we have solved). It would be interesting to understand whether (and if so which) conditions arise from analysing higher loop modular invariance. It would also be interesting to see whether there are other instances where this generalisation of discrete torsion is of significance. Finally, it would be desirable to understand the constraints of modular invariance, at least for some classes of examples, in a more conceptual fashion.

We have also proposed that, from a conformal field theoretic point of view, mirror symmetry for G_2 manifolds should be understood as a consequence of the 'mirror automorphism' of the extended G_2 algebra. This is the natural generalisation of how mirror symmetry arises for Calabi-Yau manifolds [9]. For the example considered in this paper, we have shown that this point of view precisely reproduces the symmetry proposed in [14], as well as a generalisation of the mirror symmetry found in [6, 7].

It would be interesting to understand in detail the relation of our proposal to the ideas of [19], where a duality between G_2 manifolds is conjectured using fibrewise Fourier-Mukai transformation on (co)associative fibres, and to study the Yukawa couplings they propose from a string theory point of view. It would also be interesting to understand the relation to the work of [20], that allows to establish dualities between G_2 manifolds by relating M-theory on them to various compactifications of ten-dimensional string theory. A more direct link exists to mirror symmetry for $\mathcal{N}=1$ supersymmetric flux compactifications to four dimensions considered in [21]. The half-flat manifolds appearing there give rise to G_2 manifolds upon compactification on an additional S^1. Mirror symmetry of the flux backgrounds should then induce mirror symmetry of these G_2 manifolds. Finally, it would

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be interesting to study this mirror symmetry for other G_2 manifolds for which a conformal field theory description is available [13, 22, 23, 24, 25, 26, 27, 28]. It may also be interesting to see whether there is a similar construction for Spin(8) manifolds.

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A. Discrete torsion in the $\mathbf{T^7/\mathbb{Z}_2^3}$-orbifold

In this appendix we analyse the possible generalised discrete torsion phases for the example of the orbifold $T^7/\mathbb{Z}_2^3$. We shall first analyse, for each twisted sector, how many phases can be introduced that satisfy (2.4), as well as the constraint that arises from the modular T-transformation, namely $\epsilon_f(g,g) = 1$. Once the possible phases have been determined, we shall then consider the constraints that arise from the modular S-transformation.

In the α-**twisted sector** the highest weight states are labelled by $|m,n;f\rangle_\alpha$ where m,n are the integral momentum/winding modes in the α-untwisted directions $x_{5,6,7}$ and $f = 1,\ldots,16$ label the different twist fields of lowest conformal dimension. We shall only consider the generalisation of discrete torsion where the phases depend on f, but not on any other parameters (such as the winding or momentum modes). Thus we may restrict ourselves to considering the ground state with $(m,n) = (0,0)$, leaving us with a 16-dimensional space $|0,0;f\rangle_\alpha$. The 16 twist fields are in one-to-one correspondence with the fixed points $x_j \in \{0,1/2\}$ of the action $x_j \mapsto -x_j$ of α on the first four coordinates $x_{1,2,3,4}$. The action of Γ on the label f can thus be inferred from how Γ permutes these 16 fixed points: it groups them into four orbits labelled by the fixed points in the coordinates $x_{3,4}$, each orbit consisting of the four fixed points in the coordinates $x_{1,2}$. The irreducible representations on the twist fields are therefore 4-dimensional. Labelling them by the (x_1,x_2)-coordinates of the associated fixed points as

$$|1\rangle = (x_1 = 0, x_2 = 0)\,, \qquad |2\rangle = \left(x_1 = 0, x_2 = \frac{1}{2}\right), \tag{A.1}$$

$$|3\rangle = \left(x_1 = \frac{1}{2}, x_2 = 0\right), \qquad |4\rangle = \left(x_1 = \frac{1}{2}, x_2 = \frac{1}{2}\right), \tag{A.2}$$

the generators of Γ act as follows,

$$\alpha = \mathbb{1}\,, \quad \beta = \begin{pmatrix} |1\rangle \leftrightarrow |2\rangle \\ |3\rangle \leftrightarrow |4\rangle \end{pmatrix}, \quad \gamma = \begin{pmatrix} |1\rangle \leftrightarrow |3\rangle \\ |2\rangle \leftrightarrow |4\rangle \end{pmatrix}.$$

Let

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then by possibly redefining the basis vectors by phases if necessary, we can always set

$$\beta = \left(\begin{array}{c|c} H & 0 \\ \hline 0 & H \end{array}\right) \quad \text{and} \quad \gamma = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\vartheta} \\ \hline 1 & 0 & 0 & 0 \\ 0 & e^{-i\vartheta} & 0 & 0 \end{array}\right).$$

The constraint $\beta\gamma = \gamma\beta$ implies $e^{i\vartheta} = 1$, so that

$$\gamma = \left(\begin{array}{c|c} 0 & \mathbb{1} \\ \hline \mathbb{1} & 0 \end{array}\right).$$

Hence we are able to absorb all possible phases by redefining the basis vectors. Moreover, all elements $g \in \Gamma$ with $g \neq e, \alpha$ act non-diagonally on the highest weight vectors, and thus their traces vanish.

The **β-twisted sector** is completely analogous to the α-twisted sector, except that the fixed points in the second coordinate lie now at $x_2 \in \{1/4, 3/4\}$. Upon exchanging the roles of α and β with respect to the α-twisted sector we obtain the same conclusions as above.

In the **γ-twisted sector** the 16 dimensional space spanned by $|0,0;f\rangle_\gamma$ decomposes into 8 irreducible Γ-modules each of dimension 2 and spanned by the fixed points in the x_1-plane,

$$|1\rangle = \left(x_1 = \frac{1}{4}\right), \quad |2\rangle = \left(x_1 = \frac{3}{4}\right).$$

The orbifold generators act on them as

$$\alpha = (|1\rangle \leftrightarrow |2\rangle), \quad \beta = (|1\rangle \leftrightarrow |2\rangle), \quad \gamma = \mathbb{1}.$$

By a suitable choice of basis we can always set

$$\alpha = H \quad \text{and} \quad \beta = \begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}. \tag{A.3}$$

The constraint $\alpha\beta = \beta\alpha$ then imposes $\phi \in \{0, \pi\}$, so that

$$\beta = \epsilon_\gamma H \quad \text{with} \quad \epsilon_\gamma = \pm 1. \tag{A.4}$$

Thus we have one sign degree of freedom $\epsilon_{\gamma;\tilde{f}} = \pm 1$ for each of the eight irreducible representations labelled by $\tilde{f} = 1, \ldots, 8$.

Finally, apart from e, γ there are two further group elements that act diagonally on the 16-dimensional space spanned by the $|0,0;f\rangle_\gamma$. These are $\alpha\beta$ and $\alpha\beta\gamma$, and their traces equal

$$\mathrm{Tr}_{\{|0,0;f\rangle_\gamma\}}(\alpha\beta) = \mathrm{Tr}_{\{|0,0;f\rangle_\gamma\}}(\alpha\beta\gamma) = 2\sum_{\tilde{f}=1}^{8} \epsilon_{\tilde{f}}^{(\gamma)}. \tag{A.5}$$

Next consider the **$\alpha\beta$-twisted sector**. In this (as well as all the following sectors) there is always one winding mode that takes values in $\mathbb{Z} + \frac{1}{2}$; for the case of the $\alpha\beta$-twisted sector

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this is the winding mode n_2. Since the orbifold generators invert this winding number, the ground states are now parametrised by $n_2 = \pm\frac{1}{2}$. Thus we need to look at a 32-dimensional space spanned by $|n_2 = 1/2; f\rangle_{\alpha\beta}$ and $|n_2 = -1/2; f\rangle_{\alpha\beta}$, where $f = 1, \ldots, 16$ labels the 16 twist fields corresponding to the fixed points of $x_j \mapsto -x_j$ for $j = 3, 4, 5, 6$. This space splits into 16 two-dimensional irreducible representations spanned by the two values for n_2,

$$|1\rangle = \left(n_2 = \frac{1}{2}\right), \quad |2\rangle = \left(n_2 = -\frac{1}{2}\right),$$

on which the orbifold generators act as

$$\alpha = (|1\rangle \leftrightarrow |2\rangle), \quad \beta = (|1\rangle \leftrightarrow |2\rangle), \quad \gamma = \text{diagonal}.$$

By a suitable choice of basis we can set

$$\alpha = H = \beta \quad (\text{so that } \alpha\beta = \mathbb{1}) \quad \text{and} \quad \gamma = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{pmatrix},$$

with $\epsilon_i = \pm 1$. The vanishing of the commutator of γ with α or β requires $\epsilon_1 = \epsilon_2$, and therefore

$$\gamma = \epsilon_{\alpha\beta}\mathbb{1}.$$

Incidentally, the same condition is also needed in order for the discrete torsion phases to be independent of the winding mode n_2. Each of the 16 irreducible representations has one sign, and thus we have in total

$$\text{16 signs:} \qquad \epsilon_{\alpha\beta;\tilde{f}} = \pm 1, \quad \tilde{f} = 1, \ldots, 16. \tag{A.6}$$

Apart from $e, \alpha\beta$ there are two further group elements that act diagonally on the 32-dimensional space spanned by $|n_2 = 1/2; f\rangle_{\alpha\beta}$ and $|n_2 = -1/2; f\rangle_{\alpha\beta}$. These are γ and $\alpha\beta\gamma$, and their traces over the subspace of fixed n_2 are

$$\mathrm{Tr}_{\{|n_2=1/2;f\rangle_{\alpha\beta}\}}(\gamma) = \mathrm{Tr}_{\{|n_2=1/2;f\rangle_{\alpha\beta}\}}(\alpha\beta\gamma) = \sum_{\tilde{f}=1}^{16} \epsilon_{\tilde{f}}^{(\alpha\beta)}. \tag{A.7}$$

The result for $n_2 = -1/2$ is obviously the same.

In the $\alpha\gamma$**-twisted sector** the ground states are spanned by the 16 twist fields corresponding to the fixed points in the $x_{2,4,5,7}$-plane, and the minimal winding numbers $n_1 = \pm 1/2$ along the first direction. This space splits into 8 irreducible representations of dimension 4 each, that are spanned by the fixed points in the x_2-plane and the winding numbers $n_1 = \pm 1/2$,

$$|1\rangle = \left(x_2 = 0, n_1 = -\frac{1}{2}\right), \qquad |2\rangle = \left(x_2 = 0, n_1 = \frac{1}{2}\right),$$
$$|3\rangle = \left(x_2 = \frac{1}{2}, n_1 = -\frac{1}{2}\right), \qquad |4\rangle = \left(x_2 = \frac{1}{2}, n_1 = \frac{1}{2}\right).$$

The orbifold generators act on these states as

$$\alpha = \begin{pmatrix} |1\rangle \leftrightarrow |2\rangle \\ |3\rangle \leftrightarrow |4\rangle \end{pmatrix}, \quad \beta = \begin{pmatrix} |1\rangle \leftrightarrow |4\rangle \\ |2\rangle \leftrightarrow |3\rangle \end{pmatrix}, \quad \gamma = \begin{pmatrix} |1\rangle \leftrightarrow |2\rangle \\ |3\rangle \leftrightarrow |4\rangle \end{pmatrix}.$$

By a suitable choice of basis we can set

$$\alpha = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix} = \gamma \quad (\text{so that} \quad \alpha\gamma = \mathbb{1}), \qquad \text{and} \quad \beta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & e^{i\vartheta} & 0 \\ 0 & e^{-i\vartheta} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{A.8}$$

The constraint $\alpha\beta = \beta\alpha$ then imposes $e^{i\vartheta} = 1$, so that

$$\beta = \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix}. \tag{A.9}$$

Hence we are again able to absorb all these phases into a redefinition of the basis vectors. Moreover, all elements $g \in \Gamma$, except for $g = e$ and $g = \alpha\gamma$ act non-diagonally on these highest weight vectors, and thus their traces vanish.

The $\beta\gamma$**-twisted sector** is completely analogous to the $\alpha\gamma$-twisted sector, except that the fixed points in the second coordinate now lie at $x_2 \in \{1/4, 3/4\}$. Upon exchanging the roles of α and β with respect to the $\alpha\gamma$-twisted sector we obtain the same conclusion.

Lastly we consider the $\alpha\beta\gamma$**-twisted sector**. The ground states are again spanned by the 16 fixed points in the $x_{1,4,6,7}$-plane, and the minimal winding numbers $n_2 = \pm 1/2$ in the second direction. It splits into 8 irreducible representations of dimension 4 each, that are spanned by the fixed points in the x_1-plane and the winding number $n_2 = \pm 1/2$,

$$|1\rangle = \left(x_1 = \frac{1}{4}, n_2 = -\frac{1}{2}\right), \qquad |2\rangle = \left(x_1 = \frac{1}{4}, n_2 = \frac{1}{2}\right),$$
$$|3\rangle = \left(x_1 = \frac{3}{4}, n_2 = -\frac{1}{2}\right), \qquad |4\rangle = \left(x_1 = \frac{3}{4}, n_2 = \frac{1}{2}\right).$$

The orbifold generators act on them as

$$\alpha = \begin{pmatrix} |1\rangle \leftrightarrow |4\rangle \\ |2\rangle \leftrightarrow |3\rangle \end{pmatrix}, \quad \beta = \begin{pmatrix} |1\rangle \leftrightarrow |4\rangle \\ |2\rangle \leftrightarrow |3\rangle \end{pmatrix}, \quad \gamma = \text{diagonal}.$$

By a suitable choice of basis we can set

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 & 0 & e^{i\phi} \\ 0 & 0 & e^{i\vartheta} & 0 \\ 0 & e^{-i\vartheta} & 0 & 0 \\ e^{-i\phi} & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \gamma = \begin{pmatrix} \tilde{\epsilon}_1 & 0 & 0 & 0 \\ 0 & \tilde{\epsilon}_1 & 0 & 0 \\ 0 & 0 & \tilde{\epsilon}_2 & 0 \\ 0 & 0 & 0 & \tilde{\epsilon}_2 \end{pmatrix}, \tag{A.10}$$

where we have used that the action of γ should be independent of n_2. The constraint $\alpha\beta = \beta\alpha$ then imposes $e^{i\phi} = e^{-i\phi} = \pm 1 =: \epsilon_1$ and $e^{i\vartheta} = e^{-i\vartheta} = \pm 1 =: \epsilon_2$, whereas the constraint $\alpha\gamma = \gamma\alpha$ imposes $\tilde{\epsilon}_1 = \tilde{\epsilon}_2 =: \epsilon_3$. The T-constraint $\mathbb{1} \stackrel{!}{=} \alpha\beta\gamma$ then demands $\epsilon_1 = \epsilon_3$ and $\epsilon_2 = \epsilon_3$, so that

$$\beta = \begin{pmatrix} 0 & 0 & 0 & \epsilon_1 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & \epsilon_1 & 0 & 0 \\ \epsilon_1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \gamma = \begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & \epsilon_1 & 0 & 0 \\ 0 & 0 & \epsilon_1 & 0 \\ 0 & 0 & 0 & \epsilon_1 \end{pmatrix}. \tag{A.11}$$

Each irreducible representation therefore has one sign degree of freedom $\epsilon_{\alpha\beta\gamma} = \pm 1$. On the 32-dimensional space of ground states we thus have the freedom to choose

$$\text{8 signs:} \qquad \epsilon_{\alpha\beta\gamma;\tilde{f}} = \pm 1\,, \quad \tilde{f} = 1, \ldots, 8\,. \tag{A.12}$$

Finally, apart from $e, \alpha\beta\gamma$ there are two further group elements that act diagonally on the 32-dimensional space of ground states. These are γ and $\alpha\beta$, and their traces over the subspace of fixed n_2 equals

$$\mathrm{Tr}_{\{|n_2=1/2;f\rangle_{\alpha\beta\gamma}\}}(\gamma) = \mathrm{Tr}_{\{|n_2=1/2;f\rangle_{\alpha\beta\gamma}\}}(\alpha\beta) = 2\sum_{\tilde{f}=1}^{8} \epsilon_{\alpha\beta\gamma;\tilde{f}}\,. \tag{A.13}$$

Again, the results for $n_2 = -1/2$ are obviously the same.

In summary, we therefore have 8 signs $\epsilon_{\gamma;\tilde{f}} = \pm 1$ for $\tilde{f} = 1, \ldots, 8$, in the γ-twisted sector; another 8 signs $\epsilon_{\alpha\beta\gamma;\tilde{f}} = \pm 1$ for $\tilde{f} = 1, \ldots, 8$, in the $\alpha\beta\gamma$-twisted sector; and 16 signs $\epsilon_{\alpha\beta;\tilde{f}} = \pm 1$ for $\tilde{f} = 1, \ldots, 16$, in the $\alpha\beta$-twisted sector. We are now in a position to analyse the constraint that arises from the modular S-transformation. The non-trivial conditions come from

$$\begin{aligned} &{}^{\alpha\beta}\underset{\gamma}{\square}(q(-1/\tau)) = {}^{\gamma}\underset{\alpha\beta}{\square}(q(\tau))\,, \qquad {}^{\alpha\beta\gamma}\underset{\gamma}{\square}(q(-1/\tau)) = {}^{\gamma}\underset{\alpha\beta\gamma}{\square}(q(\tau))\,, \\ &{}^{\alpha\beta}\underset{\alpha\beta\gamma}{\square}(q(-1/\tau)) = {}^{\alpha\beta\gamma}\underset{\alpha\beta}{\square}(q(\tau))\,, \end{aligned} \tag{A.14}$$

and they lead to the constraints

$$2\sum_{\tilde{f}=1}^{8} \epsilon_{\gamma;\tilde{f}} = \sum_{\tilde{f}=1}^{16} \epsilon_{\alpha\beta;\tilde{f}} = 2\sum_{\tilde{f}=1}^{8} \epsilon_{\alpha\beta\gamma;\tilde{f}}\,, \tag{A.15}$$

as claimed in the main part of the paper.

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